Electrical and Electronics Engineering
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Master Semester 2

Course
Smart grids technologies
Locational marginal prices

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Locational marginal prices

We have already introduced in lecture 4.1 the concept of **generation marginal prices**, namely the variation of the cost of a generation unit in each node as a function of the variation of the power generation of the same unit.

We can extend this concept for the so-called **locational marginal prices (LMPs), namely the variation of the overall cost of electricity of the system per variation of the demand at each (given) node**.

LMPs are a way for wholesale electric energy prices to reflect the value of electrical energy consumed **at different locations**, accounting for the patterns of load, generation, and the **physical limits of the power system**.

LMPs are computed from **Lagrange multipliers**. Let us first recall classical results on Lagrange Multipliers from the textbook:
S. Boyd and L. Vandenberghe, "Convex Optimization", Chapter 5, https://stanford.edu/~boyd/cvxbook/]

Locational marginal prices

Consider an optimization problem (P), written in Boyd and Vandenberghe's standard form, defined for $x \in \mathcal{D} \subset \mathbb{R}^n$ (where the set $\mathcal{D}$ is the *domain* of definition of (P)):

$$
\begin{aligned}
&(P): \min f(x) \\
&s.t. \\
&f_i(x) \le 0, \qquad i = 1:m \\
&h_j(x) = 0, \qquad j = 1:p \\
&x \in \mathcal{D} \subset \mathbb{R}^n
\end{aligned}
$$

The problem (P) is **feasible** if there is at least one x that satisfies the constraints. If the problem is **infeasible**, we say that the optimal value of (P) is $+\infty$.

The problem (P) is **unbounded** if it is feasible and if for any $M < 0$ there is some x such that $f(x) \le M$. If the problem (P) is **unbounded** we say that the optimal value of (P) is $-\infty$.

Locational marginal prices

The **Lagrangian** of (P) is defined for $x \in \mathcal{D}$, $\lambda_i \geq 0$ and $\nu_j \in \mathbb{R}$ as:

$$L(x, \lambda, \nu) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$$

Where λ_i and ν_j are the so-called **Lagrange multipliers** or **dual variables**.

The **dual function** $g(\lambda, \nu)$ is given by the optimal value of the **unconstrained problem** $(U(\lambda, \nu))$ defined as:

$$(U(\lambda, \nu)): \min_x L(x, \lambda, \nu)$$
$$\text{s.} t.$$
$$x \in \mathcal{D}$$

The function $g(\lambda, \nu)$ is defined on the set Γ of $\lambda, \nu \geq 0$ such that $L(x, \lambda, \nu)$ is bounded from below when x varies in $\mathcal{D}$. The **dual problem** (D) is:

$$(D): \max_{\lambda, \nu} g(\lambda, \nu)$$
$$\text{s.t.}$$
$$(\lambda, \nu) \in \Gamma$$

Locational marginal prices

Let p^* be the optimal value of (P) and d^* of (D).

(P)	(D)
$(P): min\, f(x)$ $s.t.$ $f_i(x) \leq 0, \quad i = 1{:}m$ $h_j(x) = 0, \quad j = 1{:}p$ $x \in \mathcal{D} \subset \mathbb{R}^n$	$(D): \max_{\lambda,\nu}\, g(\lambda,\nu)$ $s.t.$ $(\lambda,\nu) \in \Gamma$ $where\ g(\lambda,\nu)\ is\ given\ by$ $(U(\lambda,\nu)): \min_{x} L(x,\lambda,\nu)$ $s.t.$ $x \in \mathcal{D}$

Let us recall that, if $p^* = -\infty$, it implies (P) being unbounded and $p^* = +\infty$ means (P) is infeasible. Similarly (since (D) is a maximization problem), $d^* = -\infty$ means (D) is infeasible and $d^* = +\infty$ means (D) is unbounded.

Result 1: weak duality $d^* \leq p^*$.

This always holds and is true even in the cases where d^*, p^* are infinite. For example, Result 1 implies that if (D) is unbounded, then (P) is infeasible.

$p^* - d^*$ **is called the duality gap** and, in many cases of interest, it is 0.

Locational marginal prices

$(P): min\, f(x)$ $s.t.$ $f_i(x) \leq 0, \quad i = 1{:}\, m$ $h_j(x) = 0, \quad j = 1{:}\, p$ $x \in \mathcal{D} \subset \mathbb{R}^n$	$(D): \max_{\lambda,\nu} g(\lambda,\nu)$ $s.t.$ $(\lambda,\nu) \in \Gamma$ $where\ g(\lambda,\nu)\ is\ given\ by$ $(U(\lambda,\nu)): \min_x L(x,\lambda,\nu)$ $s.t.$ $x \in \mathcal{D}$

Definition: (x, λ) satisfy **complementary slackness** iff:
$(f_i(x) = 0$ and $\lambda_i > 0)$ or $(\lambda_i = 0)$ for $i = 1{:}\, m$

Result 2 saddle-point: let $x^* \in \mathcal{D}, \lambda_i^* \geq 0, i = 1{:}\, m,$ and $\nu_j^* \in \mathbb{R}, j = 1{:}\, p$ such that: x^* is optimal for $\left(U(\lambda^*, \nu^*)\right)$, x^* is feasible for (P) and (x^*, λ^*) satisfy complementary slackness. Then, we have that:

x^* is optimal for (P) (recall $L(x,\lambda,\nu) = f(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{j=1}^{p} \nu_j h_j(x)$)

(λ^*, ν^*) is optimal for (D)

$p^* = d^*$ meaning **strong duality**

(x^*, λ^*, ν^*) is called a **saddle-point** of the Lagrangian.

In this way we can replace the constrained problem (P) by the unconstrained problem $(U(\lambda,\nu))$.

Locational marginal prices

$(P): min\, f(x)$ $s.t.$ $f_i(x) \leq 0, \quad i = 1{:}m$ $h_j(x) = 0, \quad j = 1{:}p$ $x \in \mathcal{D} \subset \mathbb{R}^n$	$(D): \max_{\lambda,\nu} g(\lambda,\nu)$ $s.t.$ $(\lambda,\nu) \in \Gamma$ $where\ g(\lambda,\nu)\ is\ given\ by$ $(U(\lambda,\nu)): \min_x L(x,\lambda,\nu)$ $s.t.$ $x \in \mathcal{D}$

Result 3 Karush Kuhn Tucker conditions: assume (P) **is convex,** that all functions f, f_i, h_j **are differentiable** and that:

$x^* \in \mathcal{D}, \lambda_i^* \geq 0, i = 1{:}m$ and $\nu_j^* \in \mathbb{R}, j = 1{:}p$ satisfy the KKT conditions:

$$\nabla f(x^*) + \sum_i^m \lambda_i^* \nabla f_i(x^*) + \sum_i^m \nu_i^* \nabla h_j(x^*) = 0$$

x^* is feasible for (P)

(x^*, λ^*) satisfy complementary slackness

Then, we have that:

1. x^* is optimal for (P)
2. (λ^*, ν^*) is optimal for (D)
3. $p^* = d^*$ (strong duality)

KKT conditions are commonly used when (P) is convex.

Locational marginal prices

$(P): min\, f(x)$ $s.t.$ $f_i(x) \leq 0, \quad i = 1{:}m$ $h_j(x) = 0, \quad j = 1{:}p$ $x \in \mathcal{D} \subset \mathbb{R}^n$	$(D): \max_{\lambda,\nu}\; g(\lambda,\nu)$ $s.t.$ $(\lambda,\nu) \in \Gamma$ $where\ g(\lambda,\nu)\ is\ given\ by$ $(U(\lambda,\nu)): \min_x L(x,\lambda,\nu)$ $s.t.$ $x \in \mathcal{D}$

Definition: constraint qualifications by Slater's conditions:

1. there is at least one feasible point;
2. equality constraints are affine (i.e., linear equalities or inequalities);
3. inequality constraints are either affine or are satisfied strictly by at least one feasible point.

Result 4 strong duality for convex problems: assume (P) is convex and constraint qualifications hold and that x^* is optimal for (P) and (λ^*, ν^*) is optimal for (D). Then:

1. strong duality holds, i.e. $d^* = p^*$;
2. x^* is an optimal solution of $U(\lambda^*, \nu^*)$;
3. $(x,^* \ \lambda^*)$ satisfy complementary slackness.

Locational marginal prices

(P): $\min f(x)$ $s.t.$ $f_i(x) \leq 0, \quad i = 1{:}m$ $h_j(x) = 0, \quad j = 1{:}p$ $x \in \mathcal{D} \subset \mathbb{R}^n$	(D): $\max_{\lambda,\nu} g(\lambda,\nu)$ $s.t.$ $(\lambda,\nu) \in \Gamma$ $where\ g(\lambda,\nu)\ is\ given\ by$ $(U(\lambda,\nu))$: $\min_x L(x,\lambda,\nu)$ $s.t.$ $x \in \mathcal{D}$

Result 5: KKT conditions are necessary whenever strong duality holds: let us assume the following: all functions f, f_i, h_j are differentiable, that strong duality holds and that x^* is optimal for (P) and $\lambda^* \geq 0, \nu^*$ is optimal for (D). Then (x^*, λ^*, ν^*) satisfy the KKT conditions.

Corollary: let us assume that (P) is convex and satisfies constraint qualification and that all functions f, f_i, h_j are differentiable.

Then, for $x^* \in \mathcal{D}, \lambda_i^* \geq 0, i = 1{:}m$ and $\nu_j^* \in \mathbb{R}, j = 1{:}p$ we have that

x^* is optimal for (P) and λ^*, ν^* is optimal for (D) $\Leftrightarrow$ (x^*, λ^*, ν^*) satisfy the KKT conditions.

Locational marginal prices

Consider that the constraints are parameters that can be varied in the problem (P), namely:

$$P(u, r): \ \min_x f(x)$$
$$s.t.\ f_i(x) \leq u_i, \forall i = 1:m$$
$$h_j(x) = r_j, \forall j = 1:p$$

Let $c(u, r)$ be the optimal value of $P(u, r)$.

Result 6: let us assume that strong duality holds and let λ_i^*, ν_j^* be the optimal Lagrange multipliers, i.e. they are solutions of (D). If c is differentiable, then we have that:

$$\frac{\partial c}{\partial u_i} = -\lambda_i^* \text{ and } \frac{\partial c}{\partial r_j} = -\nu_j^*$$

Interpretation of the so-called **shadow price**: by adding ε_i to the i^{th}constraint (= resource i) decreases the cost by $\lambda_i \varepsilon_i$. Assume we were offered to buy an increase in resource i at a price $< \lambda_i$ per unit. We would take it since the benefit (decrease in cost) would be more than the price. Conversely, we would not take an offer to buy an increase in resource i at a price $> \lambda_i$ per unit. Thus, λ_i is the shadow price of resource i. Same with ν_j.

Locational marginal prices

Let us assume that the OPF is formulated as a convex problem (e.g. L-OPF or DC-OPF) with affine inequality constraints, so that **strong duality holds**.

$$
\begin{aligned}
&P(u, v):\ \min_x f(x) \\
&s.t. \\
&f_i(x) \le u_i, \forall i = 1{:}m \\
&h_j(x) = v_j, \forall j = 1{:}p
\end{aligned}
$$

Assume that $f(x)$ is the total cost of operation.

The locational marginal price for the load (or generator) at node h is the partial derivative of the optimal cost $c(u, v)$ with respect to the total power demand (or generation) at node h:

$$LMP_h = \frac{\partial c}{\partial P_{l_h}} = \sum_{i=1}^{m}\left(-\lambda_i^* \frac{\partial u_i}{\partial P_{l_1}}\right) + \sum_{j=1}^{p}\left(-\nu_j^* \frac{\partial v_j}{\partial P_{l_1}}\right)$$

Observation: if the total cost includes terms other than $f(x)$, they must be accounted for when computing the partial derivative (e.g. losses).

Locational marginal prices

Let us take back the DC-OPF example.

$$\min_{P_{g_2},P_{g_3}} \sum_{i=1}^{3} C_i\left(P_{g_i}\right)$$

$s.t.$

$$P_{l_1} = 22.2\,(-\theta_2) + 11.1(-\theta_3) - P_{g_1}$$
$$P_{l_2} = 22.2\,(\theta_2) + 11.1(\theta_2 - \theta_3) - P_{g_2}$$
$$P_{l_3} = 11.1(\theta_3) + 11.1\,(\theta_3 - \theta_2) - P_{g_3}$$

3 equality constraints $\Rightarrow$ 3 multipliers v_j

$$P_{g_i}^{min} \le P_{g_i} \le P_{g_i}^{max},\ i = 1,2,3$$
$$-P_{1,2}^{max} \le 22.2\,(0-\theta_2) \le P_{1,2}^{max}$$
$$-P_{1,3}^{max} \le 11.1(0-\theta_3) \le P_{1,3}^{max}$$
$$-P_{2,3}^{max} \le 11.1(\theta_2-\theta_3) \le P_{2,3}^{max}$$
$$-\pi \le \theta_i \le \pi, i = 1,2$$

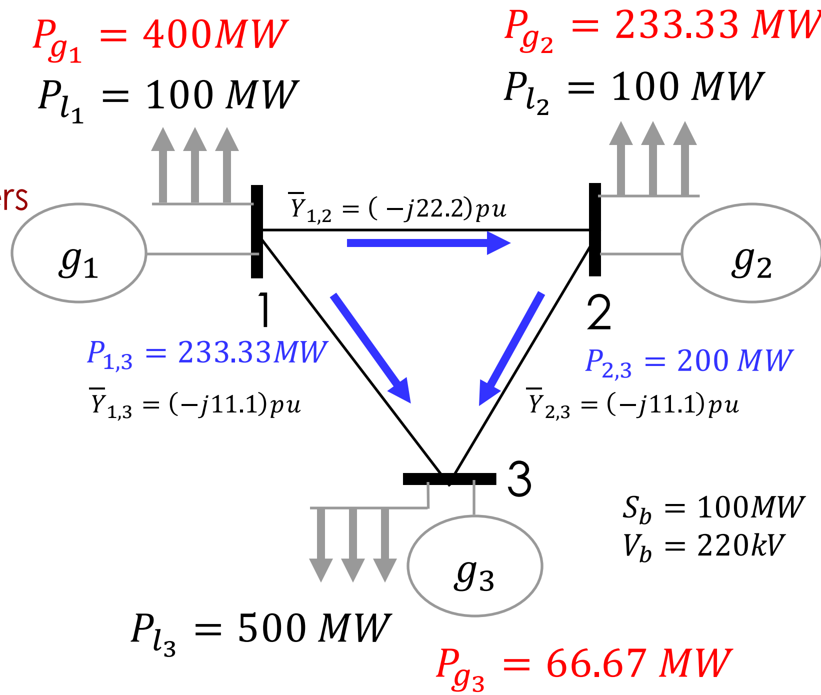
16 inequality constraints $\Rightarrow$ 16 multipliers λ_i

$$v_1 \rightarrow LMP_1 = \frac{\partial c}{\partial P_{l_1}} = 75.67\ CHF/MWh$$

$$v_2 \rightarrow LMP_2 = \frac{\partial c}{\partial P_{l_2}} = 1\ CHF/MWh$$

$$v_3 \rightarrow LMP_3 = \frac{\partial c}{\partial P_{l_3}} = 225\ CHF/MWh$$

Similarly, LMPs can be computed with respect to $P_{g_1}, P_{g_2}, P_{g_3}$



Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MW$

Locational marginal prices

It is interesting to compute the Lagrange multipliers for the **inequality constraints:**

0.0000	$\lambda_{P_{12}}$
0.0000	$\lambda_{-P_{12}}$
0.0000	$\lambda_{P_{13}}$
0.0000	$\lambda_{-P_{13}}$
37333	$\lambda_{P_{23}}$
0.0000	$\lambda_{-P_{23}}$
0.0000	λ_{θ_2}
0.0000	$\lambda_{-\theta_2}$
0.0000	λ_{θ_3}
0.0000	$\lambda_{-\theta_3}$
6066.7	$\lambda_{P_{g_1}}$
0.0000	$\lambda_{-P_{g_1}}$
0.0000	$\lambda_{P_{g_2}}$
0.0000	$\lambda_{-P_{g_2}}$
0.0000	$\lambda_{P_{g_3}}$
0.0000	$\lambda_{-P_{g_3}}$

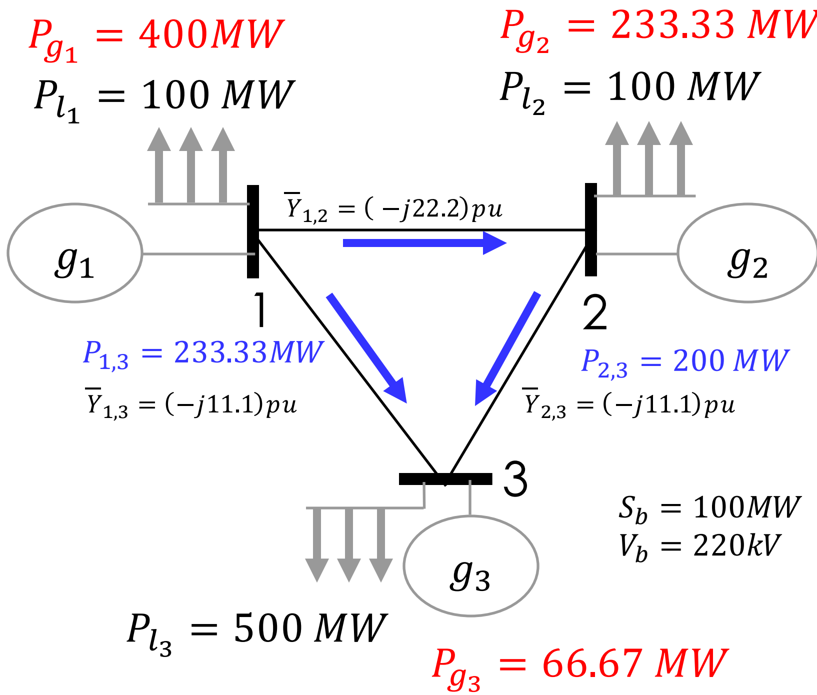
Recall the complementary slackness condition: at the optimum, when strong duality holds, an inequality constraint is satisfied with equality (the constraint is active) iff the corresponding multiplier is $\neq 0$.

Here, the constraints that are active are

$$11.1\,(\theta_2 - \theta_3) \le P_{2,3}^{max}$$

$$P_{g_1} \le 400MW$$

Generator 1 is at its limit, and so is the power flow on the line 23.



Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
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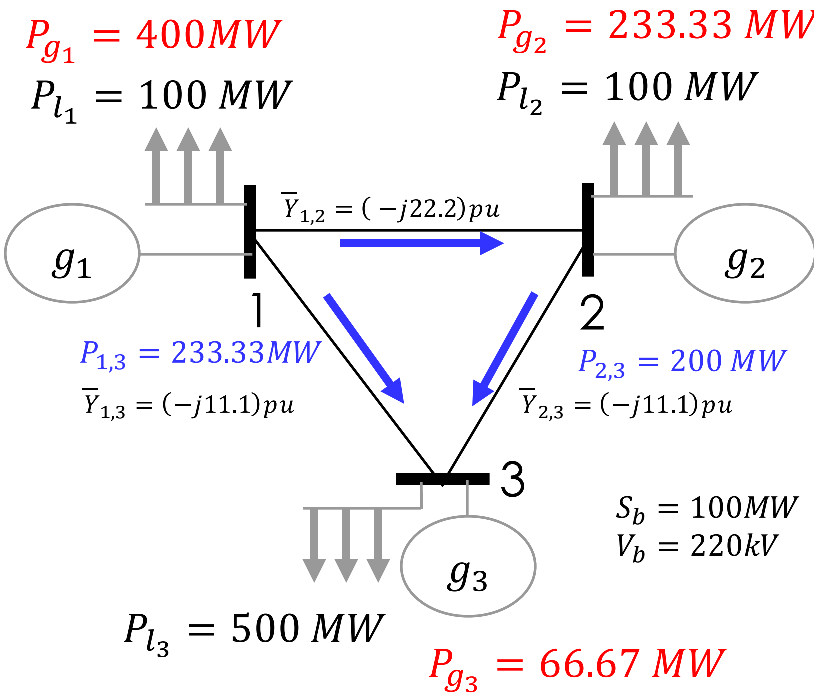
0.0000	$\lambda_{-P_{23}}$
0.0000	λ_{θ_2}
0.0000	$\lambda_{-\theta_2}$
0.0000	λ_{θ_3}
0.0000	$\lambda_{-\theta_3}$
6066.7	$\lambda_{P_{g_1}}$
0.0000	$\lambda_{-P_{g_1}}$
0.0000	$\lambda_{P_{g_2}}$
0.0000	$\lambda_{-P_{g_2}}$
0.0000	$\lambda_{P_{g_3}}$
0.0000	$\lambda_{-P_{g_3}}$

We can verify what happens if we increase the generation capacity of generator 1 by $1MW$

```
Generators (MW) =
400.0000   233.3333    66.6667
     Power Flows (MW) =
 66.6667   233.3333   200.0000
     Optimal Cost =
       2.1233e+04
```

```
Generators (MW) =
401.0000   232.6667    66.3333
     Power Flows (MW) =
 67.3333   233.6667   200.0000
     Optimal Cost =
       2.1173e+04
```

The total cost decreases by $60\ CHF$. **The marginal cost of generator** 1 **is** $60\ CHF/MWh$ **as predicted by its multiplier** (recall $S_b = 100\ MW$).



Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MW$

Locational marginal prices

It is interesting to compute the Lagrange multipliers for the **inequality constraints:**

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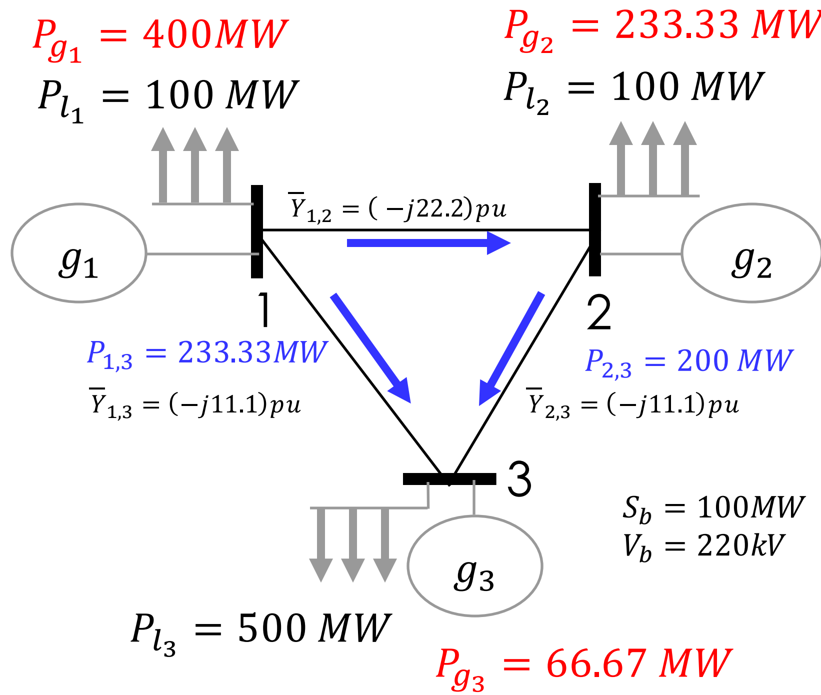
0.0000	$\lambda_{-P_{12}}$
0.0000	$\lambda_{P_{13}}$
0.0000	$\lambda_{-P_{13}}$
37333	$\lambda_{P_{23}}$
0.0000	$\lambda_{-P_{23}}$
0.0000	λ_{θ_2}
0.0000	$\lambda_{-\theta_2}$
0.0000	λ_{θ_3}
0.0000	$\lambda_{-\theta_3}$
6066.7	$\lambda_{P_{g_1}}$
0.0000	$\lambda_{-P_{g_1}}$
0.0000	$\lambda_{P_{g_2}}$
0.0000	$\lambda_{-P_{g_2}}$
0.0000	$\lambda_{P_{g_3}}$
0.0000	$\lambda_{-P_{g_3}}$

We can verify what happens if we increase the capacity of line 2 → 3 by $1MW$

```
Max branch flows (MW) =
   200    300    200
Generators (MW) =
400.0000  233.3333   66.6667
Power Flows (MW) =
 66.6667  233.3333  200.0000
Optimal Cost =
   2.1233e+04
Max branch flows (MW) =
200.0000  300.0000  201.0000
Generators (MW) =
400.0000  235.0000   65.0000
Power Flows (MW) =
 66.0000  234.0000  201.0000
Optimal Cost =
   2.0860e+04
```

The total cost decreases by $373\ CHF$ **equal to marginal cost of the power flow constraint is** $373\ CHF/MVW$**, as quantified by the value of the multiplier** (recall $S_b = 100\ MW$). **If we could buy more capacity, we would accept to pay at most** $373\ CHF/MW$**; if the price is less, we make a profit; if it is more, we lose money.**



Quantity	Value
$P_{g_i}^{min}, P_{g_i}^{max}$	$0 \div 400\ MW$
C_1, C_2, C_3	$15, 1, 225\ CHF/MWh$
$S_{12}^{max}, S_{23}^{max}, S_{31}^{max}$	$200, 200, 300\ MW$